HEAT-CONVECTION WAVES IN A SEMI-INFINITE HORIZONTAL LIQUID LAYER WITH FREE BOUNDARIES

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A solution is found for the complete problem of the propagation of small-amplitude heat-convection waves in a semi-infinite horizontal layer of a thermally compressible liquid. The waves are excited by a periodically varying temperature of a vertical side wall.

Heat-convection waves can be slightly damped in viscous, heat-conducting liquids, as was first shown in [1-3]. In an analysis of the conditions under which this phenomenon can occur, we previously showed [4] that the horizontal boundaries of the layer significantly influence the propagation of these waves. In the present paper we report a detailed study of the mechanism and features of the propagation of heat-convection waves in a semi-infinite horizontal liquid layer with free boundaries.

We consider a plane, horizontal, semi-infinite liquid layer of thickness h. A temperature T_1 is maintained at the lower boundary, while T_2 is maintained at the upper boundary. The temperature of the vertical side wall varies sinusoidally in time. The propagation of temperature oscillations within the liquid layer is a wave process. We restrict the present analysis of this process to small-amplitude waves (i.e., we assume that the amplitude of the temperature oscillations at the side wall is small). We assume the horizontal boundaries of the liquid to be free; under this assumption we can neglect terms of the second order of smallness in the equations and find an exact analytic solution of the problem, which gives a comprehensive picture of the physical situation.

We introduce a Cartesian coordinate system with x axis directed to the right along the layer and with y axis vertically upward. We use the equations for natural convection in the Boussinesq approximation:

$$\frac{\partial \Delta\Psi}{\partial t} = \Delta\Delta\Psi - \mathrm{Gr}\frac{\partial\Theta}{\partial x};$$

$$u = \frac{\partial\Psi}{\partial y};\ v = -\frac{\partial\Psi}{\partial x}; \tag{1}$$

$$\frac{\partial\Theta}{\partial t} = \frac{1}{\mathrm{Pr}}\Delta\Theta - \alpha v.$$

The boundary conditions are

$$\text{at } y=0 \quad 1-\Theta = \Psi = \frac{\partial^2\Psi}{\partial y^2} = 0;$$

$$\text{at } x=0;\ -\Theta = (1-|\alpha|)\sin\pi y \sin\omega t;\ \Psi = \frac{\partial\Psi}{\partial x} = 0. \tag{2}$$

Here the horizontal boundaries are free surfaces, while the vertical side wall is solid. We introduce the following dimensionless variables:

$$x = \frac{x'}{h};\ Y = \frac{y'}{h};\ \vec{V} = \frac{h}{\nu}\vec{V}';\ t = \frac{\nu}{h^2}t';\ \Theta = \frac{T-T_1}{(|\gamma h| + A_0)} + \alpha y. \tag{3}$$

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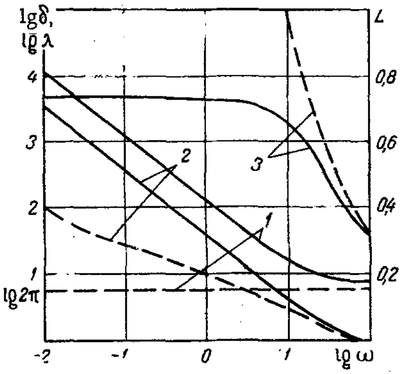

Fig. 1. Characteristics of the temperature waves in an isothermal layer. 1) Damping factor δ; 2) wavelength λ; 3) penetration depth L.

Here the parameter $\alpha = \gamma h/(|\gamma h| + A_0)$ is a measure of the ratio of the temperature-oscillation amplitude at the side wall and the temperature drop between the horizontal boundaries.

If there is no vertical temperature gradient in the layer ($T_1 = T_2$), the parameter α vanishes (this is the case of an isothermal layer). In this case, the equation for the temperature in (1) can be integrated independently of the equation of motion; i.e., in an isothermal liquid layer, small-amplitude temperature waves propagate precisely as they would in a solid plate. Because of the gravitational force, the temperature wave is accompanied by a convective wave. It is not difficult to see that in the case $\alpha = 0$ the steady-state solution of problem (1), (2) is

$$\Theta = \sin \pi y \cdot \operatorname{Im} \exp i(\omega t - kx),\ \Psi = \sin \pi y \cdot \operatorname{Im}\left\{\Psi_0 \left[\exp i(\omega t - kx) + \left(\frac{\pi + k}{\pi + k'} - 1\right) \exp i(\omega t + i\pi x) - \frac{\pi + k}{\pi + k'} \exp i(\omega t - k'x)\right]\right\}.$$

$$\Psi_0 = -\frac{ik\,\mathrm{Gr}}{\Pr(\Pr - 1)\,\omega^2};\quad k = -i\sqrt{\pi^2 + i \Pr \omega};\quad k' = -i\sqrt{\pi^2 + i\omega}. \tag{4}$$

The heat-convection waves described by Eqs. (4) are highly damped. Figure 1 shows the basic characteristics of these waves ($\lambda = 2\pi/\mathrm{Re}k$, $\delta = 2\pi \mathrm{Im}k/\mathrm{Re}k$, $L = \ln 10/\mathrm{Im}k$) as functions of the oscillation frequency ω according to the solution in (4). The dashed lines in this figure show for comparison the values of the same characteristics of a temperature wave in a solid plate with thermally insulated walls (for the one-dimensional problem), taken from [5].

Analysis of the curves in Fig. 1 shows that the damping factor δ for heat-convection waves increases rapidly with decreasing ω (curve 1) and always turns out to be larger than 2π, indicating strong damping. In a thermally insulated layer we would have $\delta = 2\pi$ for all ω. As ω decreases, the wavelength increases rapidly (curve 2), more rapidly than in a thermally insulated layer. The penetration depth L of the heat-convection waves for $\omega \leq 1$ remains nearly constant and corresponds to the propagation of these waves through about 3/4 of the height of the layer, while in a thermally insulated layer this depth increases with decreasing ω (curve 3). The reason is that the horizontal boundaries rapidly remove heat, leading to a rapid temperature equalization in the layer.

Analysis of results calculated on a computer from Eqs. (4) shows that a single-cell flow occurs near the wall in the layer as the temperature at the side wall executes a complete oscillation. The direction of the rotation in this cell reverses when the wall temperature changes sign. The intensity of this rotation increases with increasing wall temperature. There is a slight time lag (or phase shift) due to the rotary inertia. The cell is oblate; the coordinates of its center are nearly independent of the parameters of the problem if $\omega \leq 1$, being equal to 0.7h and 0.5h. It should be noted that the convection waves penetrate deeper into the layer than do the temperature waves, because the liquid motion has a cellular structure, i.e., energy is transferred not solely as a result of purely transverse shear, but also because of motion along the layer. However, because of the pronounced damping, a wave process as such (in the sense of maxima and minima varying in time and space) is essentially not observed in this situation.

Let us consider the case $\alpha \neq 0$ (in which there is a vertical temperature gradient in the layer). System (1) turns out to be coupled, which means that the temperature and transverse convective waves in the liquid interact with each other. The mechanism for this interaction is fundamentally different in the cases $\alpha < 0$ (heating from above) and $\alpha > 0$ (heating from below).

In the former case, any vertical displacement of a portion of the liquid induces an Archimedes force, which tends to restore this portion of the liquid to its previous position. In this situation there can be oscillations of portions of the liquid about an equilibrium position after some perturbation. Such oscillations are well-known as internal gravitational waves [6].

In the second case the force acting on a vertically displaced portion of the liquid is directed in the same direction as displacement. Thus, the liquid continues to move until the motion is stopped by viscosity, dissipation, and the horizontal boundaries [7]. In this case there is no oscillation of a portion of the liquid about an equilibrium position after a perturbation.

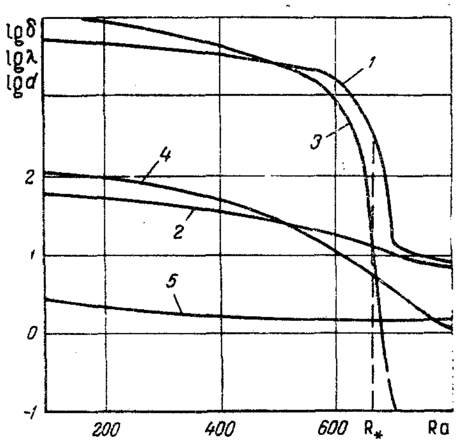

Fig. 2. Characteristics of the heat-convection waves. 1, 2) Wavelength λ for $\omega = 10^{-2}$, 1; 3, 4) δ for $\omega = 10^{-2}$, 1; 5) cell diameter d.

The mechanism for the propagation of temperature and transverse oscillations in this situation turns out to be intimately related to the cellular structure of the resulting convection, which aids the propagation of these waves as heat-convection waves [2]. A basic feature of these waves is that under certain conditions they are slightly damped in space. Below we analyze the nature and features of the propagation of heat-convection waves ($\alpha > 0$).

We seek a plane-wave solution of system (1) satisfying the boundary conditions at the horizontal boundaries:

$$[\Theta, \ \Psi] = [\Theta_0, \ \Psi_0] \sin \pi y \exp i(\omega t - kx). \tag{5}$$

Substituting (5) into (1) we find a linear, homogeneous system of algebraic equations for Θ_0, Ψ_0:

$$\begin{gathered}[i\omega + (k^2 + \pi^2)/\mathrm{Pr}]\,\Theta_0 - ik\alpha\Psi_0 = 0,\\ ik\,\mathrm{Gr}\,\Theta_0 + (i\omega + k^2 + \pi^2)(k^2 + \pi^2)\,\Psi_0 = 0.\end{gathered} \tag{6}$$

Since we are interested in nontrivial solutions of this system, we require that the determinant of this system vanish. We find a dispersion relation relating Gr, α, ω and k:

$$\left(i\omega + \frac{k^2 + \pi^2}{\mathrm{Pr}}\right)(i\omega + k^2 + \pi^2)(k^2 + \pi^2) - k^2\,\mathrm{Gr}\,\alpha = 0. \tag{7}$$

Equation (7) is a polynomial of sixth degree in k and has six complex roots. Since (7) is bicubic, only three of these roots satisfy the damping condition in the limit $x \to \infty$, $\mathrm{Im}k < 0$. A general solution of system (1) satisfying the condition at the horizontal boundaries can be written

$$\begin{gathered}\Theta = \sin \pi y \cdot \mathrm{Im} \sum_{j=1}^{3} c_j \exp i(\omega t - k_j x),\\ \Psi = \sin \pi y \cdot \mathrm{Im} \sum_{j=1}^{3} c_j \Psi_{0j} \exp i(\omega t - k_j x).\end{gathered} \tag{8}$$

The values of Ψ_{0j} are found from (6) by setting $\Theta_0 = 1$:

$$\Psi_{0j} = [i\omega + (k_j^2 + \pi^2)/\mathrm{Pr}]/ik_j\alpha.$$

It was shown previously that among the solutions in (8) there is at least one harmonic which is slightly damped under certain conditions [4], so we would expect that the superposition in (8) would represent a slightly damped heat-convection wave.

A detailed analysis of Eq. (7) shows that two of the three harmonics in (8) are slightly damped (i.e., that we have $\delta \ll 1$ under certain conditions), while one is strongly damped (i.e., in this case we always have $\delta > 2\pi$). If $x > h$, we can neglect the strongly damped harmonic; thus, if $x > h$ we can describe the slightly damped heat-convection wave as a superposition of two slightly damped modes. It turns out that the imaginary parts of the wave numbers of these modes are the same, while the real parts differ in sign and in magnitude.

To find the complex amplitudes c_j we require that conditions (2) at the vertical wall be satisfied. We then find a system of three algebraic equations with three unknowns:

$$\sum_{j=1}^{3} c_j = 1 - |\alpha|; \quad \sum_{j=1}^{3} \Psi_{0j} c_j = 0; \quad \sum_{j=1}^{3} k_j \Psi_{0j} c_j = 0. \tag{9}$$

The roots of polynomial (7) were found, and system (9) was solved on a computer. Study of the behavior of the amplitudes c_j as the parameters of the problem are varied shows that we have $c_j = (1 - |\alpha|)c_j^0$. The constants c_j^0 are independent of ω and α; $|c_j^0|$ is a very weak function of Gr, implying that there is no redistribution of energy among harmonics as Ra is changed. The amplitudes of the slightly damped modes are essentially equal in magnitude, so that in the case $x > h$ we can write the following solution for the temperature:

$$\Theta = 0{,}650\,(1 - |\alpha|)\exp(\mathrm{Im}\,k_1,\ x)\sin \pi y \sin(\omega t - l_1 x)\sin(l_2 x + \xi). \tag{10}$$

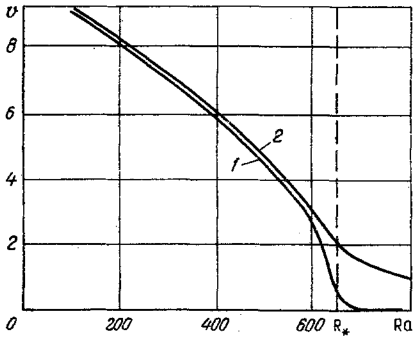

Fig. 3. Phase velocity of heat-convection waves. 1) $\omega = 10^{-2}$, 2; 2) $\omega = 1$.

where

$$l_1 = (\mathrm{Re}\, k_1 + \mathrm{Re}\, k_2)/2;\ l_2 = (\mathrm{Re}\, k_1 - \mathrm{Re}\, k_2)/2;\ \xi = \mathrm{const};$$

and k_1 and k_2 are the wave numbers of the slightly damped modes.

Equation (10) describes a damped wave with a sinusoidal amplitude propagating along the layer at a velocity $v = \omega/l_1$. Analysis of the solution for the stream function Ψ shows that there is a multicell flow in the layer, with the cell diameter given by $d = \pi/l_2$. The intensity of the rotation in these cells falls off away from the side wall, so we can conclude that the sinusoidal nature of the amplitude in (10) is a result of the multicell nature of the resulting flow. For all reasonable parameter values, the quantity l_2 turns out to be larger than l_1; i.e., the wavelength of the traveling wave, $\lambda = 2\pi/l_1$, is always larger than the cell diameter d. Figure 2 shows the wavelength λ, the damping factor δ, and the diameter d as functions of Ra for various values of ω. Analysis of these curves shows that the cell diameter d varies very slightly (from 2.5h to 1.3h) with increasing Ra and is nearly independent of ω (curve 5). This behavior is evidence that the cellular flow in the heat-convection wave is intimately related to the layer height h. The wavelength of the heat-convection wave falls off with increasing Ra; at small ω this decrease is most pronounced at Rayleigh numbers approximately equal to the critical value R_* (curve 1). At frequencies $10^{-2} < \omega < 1$ the curves of λ as a function of Ra lie between curves 1 and 2. It should be noted that in this range of Rayleigh numbers the wavelength of the heat-convection waves is much larger than the layer height.

Analysis of the damping factor $\delta = \lambda/L$ as a function of the Rayleigh number shows that δ falls off more rapidly with increasing Rayleigh number than does the wavelength; this behavior is a result of the increase in the penetration depth L. Curves 3 and 4 show the extreme dependences of δ at frequencies $10^{-2} < \omega < 1$. At sufficiently small frequencies ω the condition $\delta \ll 1$ can be satisfied if $\mathrm{Ra} > R_*$ (curve 3). Although the wavelengths turn out to be much larger than the layer height ($\lambda \simeq 10h$) in this case, the penetration depth L remains much larger than λ.

At Rayleigh numbers $\mathrm{Ra} < R_*$ the damping factor is $\delta > 1$; i.e., the damping occurs over a distance shorter than the wavelength. However, we note that the wavelength is quite large in this region and that the penetration depth L with Ra near R_* turns out to be large in comparison with the cell diameter. As a result, in the case of low-frequency temperature oscillations at the side wall the heat-convection wave in the layer is approximately a standing wave, described by Eq. (10) with $l_1 = 0$. The penetration depth for such a wave turns out to be large at sufficiently small values of ω and at Rayleigh numbers Ra approximately equal to R_*. In calculating the phase velocity we should recall that l_1 is not identically equal to zero. The propagation velocity v remains finite, since l_1 and ω turn out to be of the same order of smallness. We see from Fig. 3 that the velocity v decreases with increasing Ra because of the decrease of the wavelength. At $\mathrm{Ra} < R_*$, the velocity v is nearly independent of the frequency ω.

We attribute the important change in the nature of the heat-convection waves at the transition through R_* to the circumstance that in the range $\mathrm{Ra} > R_*$ there is pronounced steady-state cellular convection in the layer; i.e., in this case we can write the solution of problem (1)-(2) as

$$\Psi = \Psi_0(xy) + \Psi_t(xyt),\ \Theta = \Theta_0(xy) + \Theta_t(xyt). \tag{11}$$

Here Ψ_0, Θ_0 are the steady-state solution of problem (1)-(2) with boundary condition $\Theta_0(0, y) = 0$, while Ψ_t, Θ_t is a transient solution. In the range $\mathrm{Ra} < R_*$ we have $\Psi_0 = \Theta_0 \equiv 0$. In the range $\mathrm{Ra} > R_*$ the solution Ψ_0, Θ_0 describes steady-state cellular convection, so that the heat-convection waves also propagate against a background of this convection, which contributes to the slight damping of these waves. The amplitude of the steady-state convection is governed, not by the amplitude of the oscillations at the side wall, but by the difference $((\mathrm{Ra} - R_*)/\mathrm{Ra})^{1/2}$, so that at sufficiently large values of Ra we must take into account the nonlinear convective terms in system (1), although the amplitude of the heat-convection waves is small. In the present paper we restrict the discussion to a study of the linear problem; i.e., we ignore Rayleigh numbers much larger than R_*. Numerical calculations carried out over a broad range of Ra in the layer with solid walls confirm the qualitative picture found in this linear approximation. The results will be published in the near future.

NOTATION

T	is the temperature;
h	is the layer height;
x, y	are the Cartesian coordinates;
ω	is the oscillation frequency;
ν	is the kinematic viscosity;
a	is the thermal diffusivity;
β	is the coefficient of thermal expansion;
R_*	is the critical Rayleigh number, giving the threshold for the convective instability of a layer heated from below;
Ψ	is the stream function;
u, v	are the velocity components;
Θ	is the deviation of the temperature from its value at the equilibrium position;
$Gr = \beta g h^3(\lvert\gamma h\rvert + A_0)/\nu^2$	is the Grashof number;
$Pr = \nu/a$	is the Prandtl number;
A_0	is the maximum amplitude of the temperature oscillations at the side wall;
$\gamma = (T_1 - T_2)/h$	is the vertical temperature gradient.

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